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Expo. Math. 27 (2009) 37 – 53

EXPOSITIONES MATHEMATICAE

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# $u$-Invariants for forms of higher degree

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Received 12 February 2008; received in revised form 1 May 2008

**Abstract**

Both a general and a diagonal $u$-invariant for forms of higher degree are defined, generalizing the $u$-invariant of quadratic forms. We give a survey of both old and new results on these $u$-invariants.


***MSC 2000:*** primary 11E76 secondary 11E04 12E05

***Keywords:*** Forms of higher degree; Diagonal forms; Non-trivial zeros; $u$-Invariant; $p$-Adic zeros

## Introduction

Let $k$ be a field of characteristic 0 or greater than $d$. The $u$-invariant (*of degree* $d$) of $k$ is defined as $u(d,k) = \sup\{\dim_k \varphi\}$, where $\varphi$ ranges over all the anisotropic forms of degree $d$ over $k$. Since for $d \geqslant 3$ not all forms are diagonal, we also define the *diagonal $u$-invariant* (*of degree* $d$) over $k$ as $u_{\mathrm{diag}}(d,k) = \sup\{\dim \varphi\}$, where $\varphi$ ranges over all the anisotropic diagonal forms over $k$. Obviously, $u_{\mathrm{diag}}(d,k) \leqslant u(d,k)$. For $d=2$, the definitions of $u_{\mathrm{diag}}(d,k)$ and $u(d,k)$ coincide and correspond to the classical $u$-invariant of quadratic forms over $k$.

For an algebraically closed field $k$, $|k^\times/k^{\times d}| = 1$ and hence each form of degree $d$ and dimension greater one over $k$ is isotropic. This means

$$u_{\mathrm{diag}}(d,k) = u(d,k) = 1.$$

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0723-0869/$ - see front matter 
doi:10.1016/j.exmath.2008.07.004

For formally real fields $k$, however, the value of the $u$-invariant is infinite for even $d$: since $-1 \notin \sum k^2$, also $-1 \notin \sum k^d$ for any even $d$. Thus the form $m \times \langle 1 \rangle$ of degree $d$ given by the homogeneous polynomial $F(x_1, \dots, x_m) = x_1^d + \dots + x_m^d$ is anisotropic for each integer $m$, implying

$$u_{\mathrm{diag}}(d, k) = u(d, k) = \infty$$

for even $d$. Until Merkurjev's celebrated results [M1,M2] very little was known about which values the $u$-invariant of quadratic forms over a nonreal field can take. The situation for forms of higher degree is similar. We present a selection of results and some new observations, putting special emphasis on $u$-invariants of finite fields, rational function fields and discretely valued fields. One should point out that especially $p$-adic fields attracted a lot of attention due to Artin's conjecture [A] on the $u$-invariant, which states that $u(d, \mathbb{Q}_p) \leqslant d^2$. Although Artin's conjecture is believed to be true for forms of prime degree, it does not hold in general. The conjecture was shown to be false by Terjanian [T] in 1966, who produced an 18-dimensional anisotropic form as a counterexample for the case $d = 4$ and $p = 2$. Nonetheless, upper bounds for $u(d, \mathbb{Q}_p)$ are a focal point of many investigations up to this day.

## 1. Preliminaries

**1.1.** Let $k$ be a field of characteristic 0 or greater than $d$. A *d-linear form* over $k$ is a $k$-multilinear map $\theta: V \times \dots \times V \to k$ ($d$-copies) on a finite-dimensional vector space $V$ over $k$ which is *symmetric*, i.e. $\theta(v_1, \dots, v_d)$ is invariant under all permutations of its variables. A *form of degree d* over $k$ is a map $\varphi: V \to k$ on a finite-dimensional vector space $V$ over $k$ such that $\varphi(av) = a^d \varphi(v)$ for all $a \in k$, $v \in V$ and such that the map $\theta: V \times \dots \times V \to k$ defined by

$$\theta(v_1, \dots, v_d) = \frac{1}{d!} \sum_{1 \leqslant i_1 < \dots < i_l \leqslant d} (-1)^{d-l} \varphi(v_{i_1} + \dots + v_{i_l})$$

($1 \leqslant l \leqslant d$) is a $d$-linear form over $k$. By fixing a basis $\{e_1, \dots, e_n\}$ of $V$, any form $\varphi$ of degree $d$ can be viewed as a homogeneous polynomial of degree $d$ in $n = \dim V$ variables $x_1, \dots, x_n$ via $\varphi(x_1, \dots, x_n) = \varphi(x_1 e_1 + \dots + x_n e_n)$ and, vice versa, any homogeneous polynomial of degree $d$ in $n$ variables over $k$ is a form of degree $d$ and dimension $n$ over $k$. Any $d$-linear form $\theta: V \times \dots \times V \to k$ induces a form $\varphi: V \to k$ of degree $d$ via $\varphi(v) = \theta(v, \dots, v)$. We can identify $d$-linear forms and forms of degree $d$ with the help of the obvious correspondence.

**1.2.** Two $d$-linear spaces $(V_i, \theta_i)$, $i = 1, 2$, are called *isomorphic* (written $(V_1, \theta_1) \cong (V_2, \theta_2)$ or just $\theta_1 \cong \theta_2$) if there exists a bijective linear map $f: V_1 \to V_2$ such that $\theta_2(f(v_1), \dots, f(v_d)) = \theta_1(v_1, \dots, v_d)$ for all $v_1, \dots, v_d \in V_1$. A $d$-linear space $(V, \theta)$ (or the $d$-linear form $\theta$) is called *nondegenerate* if $v = 0$ is the only vector such that $\theta(v, v_2, \dots, v_d) = 0$ for all $v_i \in V$. A form of degree $d$ is called *nondegenerate* if its associated $d$-linear form is nondegenerate. We will only study nondegenerate forms.

If we can write a form $\varphi$ of degree $d$ in the form $a_1 x_1^d + \dots + a_m x_m^d$ we use the notation $\varphi = \langle a_1, \dots, a_n \rangle$ and call the form $\varphi$ *diagonal*. A diagonal form $\varphi = \langle a_1, \dots, a_n \rangle$ is nondegenerate if and only if $a_i \in k^\times$ for all $1 \leqslant i \leqslant n$. For char $k \neq 2$, every quadratic form is isomorphic to a diagonal form. This is not true any more for forms of higher degree.

**1.3.** The *orthogonal sum* $(V_1, \theta_1) \perp (V_2, \theta_2)$ of two $d$-linear spaces $(V_i, \theta_i)$, $i = 1, 2$, is defined to be the $k$-vector space $V_1 \oplus V_2$ together with the $d$-linear form

$$(\theta_1 \perp \theta_2)(u_1 + v_1, \dots, u_d + v_d) = \theta_1(u_1, \dots, u_d) + \theta_2(v_1, \dots, v_d)$$

($u_i \in V_1$, $v_i \in V_2$). The *tensor product* $(V_1, \theta_1) \otimes (V_2, \theta_2)$ is the $k$-vector space $V_1 \otimes V_2$ together with the $d$-linear form [H-P]

$$(\theta_1 \otimes \theta_2)(u_1 \otimes v_1, \dots, u_d \otimes v_d) = \theta_1(u_1, \dots, u_d) \cdot \theta_2(v_1, \dots, v_d).$$

A $d$-linear space $(V, \theta)$ is called *decomposable* if $(V, \theta) \cong (V, \theta_1) \perp (V, \theta_2)$ for two non-zero $d$-linear spaces $(V, \theta_i)$, $i = 1, 2$. A non-zero $d$-linear space $(V, \theta)$ is called *indecomposable* if it is not decomposable and *absolutely indecomposable*, if it stays indecomposable under each algebraic field extension.

**1.4.** Let $k$ be a field and $v: k \to \Gamma \cup \infty$ a *valuation* of $k$, that is $\Gamma$ is a totally ordered additive abelian group, $\infty$ a symbol that does not lie in $\Gamma$, and $a < \infty$, $a + \infty = \infty + a = \infty + \infty = \infty$ for all $a \in \Gamma$. Moreover, $v(x) = \infty$ iff $x = 0$, $v(xy) = v(x) + v(y)$ and $v(x + y) \geqslant \min\{v(x), v(y)\}$ for all $x, y \in k$. $(k, v)$ is called a *valued field* and $\Gamma$ the *value group* of $v$. $v$ is called *trivial* if $\Gamma = 0$. We will only deal with non-trivial valuations. The subring $R = \{x \in k | v(x) \geqslant 0\}$ of $k$ is called the *valuation ring* of $v$ and its only maximal ideal $\{x \in k | v(x) > 0\}$ is denoted $m$. We have $k = \mathrm{Quot}(R)$. $\bar{k} = R/m$ is called the *residue field* of $v$. A *discrete valuation* is a non-trivial valuation $v$ with value group $\Gamma = \mathbb{Z}$. For a discretely valued field $(k, v)$, an element $\pi \in R$ is called a *uniformizing parameter* if $v(\pi) = 1$. We have $m = (\pi)$ (cf. [R]).

## 2. Some estimates on higher $u$-invariants

**2.1.** Similar to the well-known case $d = 2$, the invariants $u(d, k)$ and $u_{\mathrm{diag}}(d, k)$ can be characterized as follows: $u_{\mathrm{diag}}(d, k)$ is the smallest integer $n$ such that all diagonal forms of degree $d$ over $k$ of dimension greater than $n$ are isotropic, and $u(d, k)$ is the smallest integer $n$ such that all forms of degree $d$ over $k$ of dimension greater than $n$ are isotropic.

If $u = u(d, k)$ is finite, then each anisotropic form $(V, \varphi)$ of degree $d$ over $k$ and dimension $u$ is *universal*, i.e. for each $a \in k^\times$ there is an $x \in V$ such that $\varphi(x) = a$. If $u = u_{\mathrm{diag}}(d, k)$ is finite, then each diagonal anisotropic form of degree $d$ over $k$ and dimension $u$ is universal. Moreover, we have

$$u_{\mathrm{diag}}(d, k) \leqslant \min\{n | \text{all forms of degree } d \text{ over } k \text{ of dimension} \geqslant n \text{ are universal}\}$$

with the understanding that the "minimum" of an empty set of integers is the symbol $\infty$.

Apart from the obvious relation that $u_{\mathrm{diag}}(d, k) \leqslant u(d, k)$, we know that the $u$-invariants

$$u(2, k), u(3, k), \dots, u(d, k)$$

must be finite, provided that the diagonal $u$-invariants

$$u_{\mathrm{diag}}(2, k), u_{\mathrm{diag}}(3, k), \dots, u_{\mathrm{diag}}(d, k)$$

are finite Brauer [G, (8.1)]. Estimates for $u(d,k)$ using $u(s,k)$ for $s=2,\dots,d-1$ and/or $u_{\mathrm{diag}}(d,k)$ are given by Leep–Schmidt [Le-S, Theorem 2] and others.

Some simple lower bounds can be obtained as follows:

**Lemma 1.** (i) $u(3,k)\neq 2$.

(ii) *For every integer n we have* $u(d^n,k)\geqslant u(d,k)^n$.

(iii) *If* $u(d,k)=u$ *then* $u(md,k)\geqslant u$ *for each integer* $m>1$.

**Proof.** (i) Suppose that $u(3,k)=2$. This means there exists a homogeneous polynomial $f(x,y)=a_0x^3+a_1x^2y+a_2xy^2+a_3y^3$ in two variables over $k$ which is anisotropic. Therefore there exists an irreducible polynomial $f(t)=a_0+a_1t+a_2t^2+a_3t^3$ of degree 3 over $k$, because $f(x,y)$ is isotropic over $k$ if and only if $f(t)$ has a root in $k$. (Note that $f(a,b)=a^3f(b/a)$ for $a\neq 0$ and that $f(0,b)=a_3b^3$.) Hence there is a field extension $l/k$ of degree 3. Let $n_{l/k}$ be its anisotropic norm and let $\{v_1,v_2,v_3\}$ be a $k$-basis of $l$. Then the form $g(x_1,x_2,x_3)=n_{l/l}(x_1v_1+x_2v_2+x_3v_3)$ is anisotropic and thus $u(3,k)\geqslant 3$, a contradiction.

(ii) Let $f(X)=f(x_1,\dots,x_u)$ be an anisotropic form of degree $d$ in $u$ variables over $k$. Let $f_2(X_1,\dots,X_u)=f(f(X_1),\dots,f(X_u))$ where each $X_j$ is a different set of $u$ variables. Since $f_2$ is an anisotropic form of degree $d^2$ in $u^2$ variables, we have $u(d^2,k)\geqslant u^2$. Repeating this argument yields the assertion (see also [S, p. 99, 15.7]).

(iii) Suppose that $\varphi_1$ is an anisotropic form of degree $d$ over $k$. Then $\varphi$ defined via $\varphi(z_1,\dots,z_u)=\varphi_1(z_1,\dots,z_u)^m$ is anisotropic of degree *md*. $\square$

Let $\varphi$ be a form of degree $d$ on a $k$-vector space $V$. Write $D(\varphi)=\{a\in k^\times \mid \varphi(x)=a$ for some $x\in V\}$ for the set of non-zero elements represented by $\varphi$. We sometimes work with the factor group $D(\varphi)/k^{\times d}$. When we regard $D(\varphi)$ as a subset of $k^\times/k^{\times d}$, we will also write $D(\varphi)$ instead of writing $D(\varphi)/k^{\times d}$, abusing notation.

**Remark 1** (*cf. Lam [La, p. 14] for* $d=2$). Let $\varphi$ be a form of degree $d$ over $k$ and $a\in k^\times$.

(i) If $a\in D(\varphi)$ then $\varphi\perp\langle -a\rangle$ is isotropic.
(ii) If $\varphi$ is anisotropic and $\varphi\perp\langle -a\rangle$ isotropic then $a\in D(\varphi)$.

**Lemma 2.** *Let* $\varphi$ *be a form of degree d over k.*

(i) *If* $\varphi$ *is anisotropic over k, then* $\varphi$ *remains anisotropic over* $k(t)$.
(ii) (*for* $d=2$, *cf.* [La, p. 256]). *If* $\varphi$ *is an anisotropic form of degree d over k, then*

$$D(\varphi_{k(t)})\cap k=D(\varphi).$$

**Proof.** (i) Suppose that $\varphi$ is isotropic over $k(t)$, then there are $f_i(t)/g_i(t)\in k(t)$, not all of them zero, such that $\varphi(f_1(t)/g_1(t),\dots,f_n(t)/g_n(t))=0$. Clearing denominators we can assume without loss of generality that there are $f_i(t)\in k[t]$, not all of them zero, such that $\varphi(f_1(t),\dots,f_n(t))=0$. Changing these if necessary assume moreover that $t$ does not divide all of them. Put $t=0$ to obtain $\varphi(f_1(0),\dots,f_n(0))=0$ with not all of the $f_i(0)$ zero, so we found an isotropic vector of $\varphi$ over $k$.

(ii) Let $a \in D(\varphi_{k(t)}) \cap k$, then $\varphi_{k(t)} \perp \langle -a \rangle$ is isotropic over $k(t)$ and therefore so is $\varphi \perp \langle -a \rangle$ over $k$ by (i). This implies $a \in D(\varphi)$ by Remark 1. $\square$

**Lemma 3.** *Let $K = k(t)$ be the rational function field or let $K = k((t))$ be the Laurent series field over $k$ and let $\varphi$ be a form of degree $d$ over $k$. Then the form*

$$\langle 1, t, t^2, \ldots, t^{d-1} \rangle \otimes \varphi = \varphi \perp t\varphi \perp t^2\varphi \perp \cdots \perp t^{d-1}\varphi$$

*of degree $d$ is anisotropic over $K$ if and only if $\varphi$ is anisotropic over $k$.*

**Proof.** Let $K = k(t)$ be the rational function field. Let $\varphi$ be a form over $k$ in $n$ variables. Assume that the form $\varphi \perp t\varphi \perp t^2\varphi \perp \cdots \perp t^{d-1}\varphi$ of degree $d$ is isotropic over $k(t)$. Then there exist polynomials $f_i(t) \in k[t]$, not all of them zero, such that

$$\begin{aligned}&\varphi(f_1(t), \ldots, f_n(t)) + t\varphi(f_{n+1}(t), \ldots, f_{2n}(t)) + t^2\varphi(f_{2n+1}(t), \ldots, f_{3n}(t))\\ &\quad + \cdots + t^{d-1}\varphi(f_{(d-1)n+1}(t), \ldots, f_{dn}(t)) = 0.\end{aligned} \tag{1}$$

Assume additionally that the value for $\sum_{i=1}^{n} \deg f_i(t)$ is minimal. Plugging in $t = 0$ shows that $\varphi(f_1(0), \ldots, f_n(0)) = 0$ and thus $f_i(0) = 0$ for all $i$, $1 \leqslant i \leqslant n$, because $\varphi$ is anisotropic over $k$. Hence $f_i(t) = tg_i(t)$ for $1 \leqslant i \leqslant n$. Substituting this into (1) and cancelling $t$ yields a version of (1) with decreased $\sum_{i=1}^{n} \deg f_i(t)$, a contradiction.

The same argument applies if $K = k((t))$ is the Laurent series field. $\square$

Indeed, using the above notation a similar argument shows that given forms $\varphi_i$ of degree $d$ over $k$, the form

$$\varphi_1 \perp t\varphi_2 \perp t^2\varphi_3 \perp \cdots \perp t^{d-1}\varphi_d$$

of degree $d$ is anisotropic over $k(t)$ if and only if the forms $\varphi_i$ are anisotropic over $k$ for all $i$.

**Remark 2.** Lemma 3 can be also proved using [Mo, Theorem 2.5]. If the form $\varphi$ in Lemma 3 has dimension $d^i$, Lemma 3 is a special case of [G, 4.10] (cf. Theorem 2).

As a consequence of Lemmas 2 and 3 we obtain:

**Corollary 1.** *Let $K = k(t)$ be the rational function field or let $K = k((t))$ be the Laurent series field over $k$. Then*

$$u(d, K) \geqslant du(d, k),$$

$$u_{\text{diag}}(d, K) \geqslant du_{\text{diag}}(d, k),$$

*and $s_d(k) = s_d(k(t))$.*

A field $k$ is called a *$C_i$-field*, if every form of degree $d$ over $k$ in at least $d^i + 1$ variables is isotropic [G], that is we have

$$u(d, k) \leqslant d^i$$

in this case. The rational function field in one variable $k(t)$ is a $C_{i+1}$-field. If $k$ is a $C_i$-field, then so is each algebraic field extension $l$ over $k$, hence also $u(d,l)\leqslant d^i$. Moreover, $k=\mathbb{C}(t_1,\dots t_n)$ is a $C_n$-field [S, p. 97]. These results of Tsen and Lang motivated much of the later work done in this direction ([S, p. 97], cf. also [G]).

**Example 1.** (i) Let $k=k_0(t_1,\dots,t_n)$ with $k_0$ a field of characteristic 0 or $>d$ and with $t_1,\dots,t_n$ independent indeterminates over $k$. By induction on $n$ (using the above results) the form $\langle 1,t_1,t_1^2,\dots,t_1^{d-1}\rangle\otimes\cdots\otimes\langle 1,t_n,\dots,t_n^{d-1}\rangle$ of degree $d$ and dimension $d^n$ is anisotropic over $k$. Hence

$$u(d,k_0(t_1,\dots,t_n))\geqslant u_{\mathrm{diag}}(d,k_0(t_1,\dots,t_n))\geqslant d^n.$$

Since $k=\mathbb{C}(t_1,\dots t_n)$ is a $C_n$-field, it follows that $u(d,\mathbb{C}(t_1,\dots,t_n))\leqslant d^n$ and $u_{\mathrm{diag}}(d,\mathbb{C}(t_1,\dots,t_n))\leqslant d^n$. We conclude that

$$u(d,\mathbb{C}(t_1,\dots,t_n))=u_{\mathrm{diag}}(d,\mathbb{C}(t_1,\dots,t_n))=d^n.$$

Thus every power of $d$ is the $u(d,k)$-invariant (resp. the $u_{\mathrm{diag}}(d,k)$-invariant) of some suitable field $k$.

(ii) Let $k_0$ be a field of characteristic 0 or $>3$, such that there exists a central division algebra of degree 3 over $k_0$. Then $u(3,k_0)\geqslant 9$ and $u(3,k_0(t))\geqslant 3\times 9=27$ by Corollary 1. Indeed, there also exists an Albert division algebra over $k_0(t)$ [KMRT, p .531]. Its norm is an absolutely indecomposable anisotropic cubic form over $k_0(t)$ of dimension 27. Thus the bounds $u(3,k_0)\geqslant 9$ and $u(3,k_0(t))\geqslant 27$ are best possible.

**Remark 3.** Take the field $\mathbb{Q}$. For every prime $p$, the form $\langle 1,p,\dots,p^{d-1}\rangle$ of degree $d$ is anisotropic over $\mathbb{Q}$ [I-R, p. 150]. Hence

$$u_{\mathrm{diag}}(d,\mathbb{Q})\geqslant d.$$

If $K$ is an algebraic number field of finite degree over $\mathbb{Q}$, and if $d$ is an odd positive integer, then there is an integer $M(K,d)$ such that for $n>M(K,d)$, any form of degree $d$ in $n$ variables over $K$ is isotropic [G].

For $d=3$, it is easy to see that there are nondegenerate cubic forms in 9 variables which are anisotropic (e.g., the norm form of any central division algebra of degree 3 over $\mathbb{Q}$), so that as explained in Example 1 (ii), the lower bounds

$$u(3,\mathbb{Q})\geqslant 9 \quad\text{and}\quad u(3,\mathbb{Q}(t))\geqslant 27$$

are best possible. Heath-Brown [H-B1] showed that each nonsingular cubic form over $\mathbb{Q}$ of dimension $\geqslant 10$ is isotropic using the "circle method", which was already applied by Davenport [D], who showed that any (even degenerate) cubic form over $\mathbb{Q}$ of dimension $\geqslant 16$ is isotropic.

**2.2.** The $d$th *level* (also called *power Stufe* in [P-A-R]) $s_d(k)$ of $k$ is the least positive integer $s$ for which the equation

$$-1=a_1^d+\cdots+a_s^d$$

is solvable in $k$. If there is no such integer, define $s_d(k)=\infty$. We have $s_d(k)\leqslant u_{\mathrm{diag}}(d,k)$. In case $d$ is odd, $s_d(k)=1$. For $d=2$ the level of a field of characteristic not 2 was studied among others by Pfister [Pf2], who proved that $s_2(k)$, if finite, is always a power of 2.

For $d\geqslant 3$, however, the $d$th level is not always a power of $d$: Parnami et al. [P-A-R] proved that $s_4(\mathbb{Q}(\sqrt{-m}))=15$ for $m\equiv 7 \bmod 8$, and $s_4(\mathbb{Q}(\sqrt{-2}))=6$.

Higher levels have been studied extensively for finite fields. The value of $s_d(\mathbb{F}_q)$ was computed by Pall and Rajwade [P-R] for $d\leqslant 10$, using cyclotomic numbers and Jacobi sums for finite fields. Amice–Kahn [Am-K] studied $s_d(\mathbb{F}_q)$ for $d$ a power of 2. In 1999, the $d$th levels of $\mathbb{F}_p$, $p$ a prime, were computed by Becker–Canales [B-C]. They proved that $s_d(\mathbb{F}_p)$ is determined by a formula involving the coefficients of the Gauss period equation of degree $d$ associated with $p$. Their results gave new insight into the behaviour of $d$th levels of $p$-adic fields $\mathbb{Q}_p$, if $p\geqslant 2$.

Let $k$ be an arbitrary field. It is clear that every element in $k$ can be written as a sum of $n$ $d$th powers provided the form $n\times\langle 1\rangle=\langle 1,\dots,1\rangle$ of degree $d$ is universal for some $n$. For a field $k$ of finite $d$th level $s=s_d(k)$, the form $s\times\langle 1\rangle=\langle 1,\dots,1\rangle$ of degree $d$ is anisotropic over $k$. We conclude:

**Proposition 1.** *Let $k$ be a field such that $s=s_d(k)=u_{\mathrm{diag}}(d,k)$. Then every element in $k$ can be written as a sum of $s$ $d$th powers.*

## 3. Demyanov's theorem for forms of higher degree

For fields which have finite $d$th level, the diagonal $u$-invariant can be bounded above, as follows from the next proposition:

**Proposition 2** (*for $d=2$, cf. Lam [La, p. 317] or Scharlau [S, p. 104]*). *Let $k$ be a field with finite $d$th level $s_d(k)=s<\infty$. Suppose that $\varphi$ is a form of degree $d$ over $k$ such that $\varphi\perp\langle a\rangle$ is anisotropic for some $a\in k^\times$. Then $D(\varphi)\subsetneqq D(\varphi\perp\langle a\rangle)$.*

**Proof.** Assume instead that $D(\varphi)=D(\varphi\perp\langle a\rangle)$, then, in particular, we have $a\in D(\varphi)$. Write $-1=e_1^d+\cdots+e_s^d$ with suitable $e_i\in k^\times$. We claim that $a(e_1^d+\cdots+e_i^d)\in D(\varphi)$ for $1\leqslant i\leqslant s$: if $i=1$, obviously $e_1^d a\in D(\varphi)$. Suppose we have proved the assertion for $i-1$, i.e. $a(e_1^d+\cdots+e_{i-1}^d)\in D(\varphi)$. Then $a(e_1^d+\cdots+e_{i-1}^d)+ae_i^d\in D(\varphi\perp\langle a\rangle)=D(\varphi)$. In particular, $-a=a(e_1^d+\cdots+e_s^d)\in D(\varphi)$, hence $-a=\varphi(x)$, which implies that $\varphi\perp\langle a\rangle$ is isotropic, contradicting our assumption. $\square$

**Theorem 1** (*Leep et al. [Le-Y1, 2.3] or Demyanov [De2]*). *Let $k$ be a field of arbitrary characteristic with finite $d$th level $s_d(k)<\infty$. Then*

$$u_{\mathrm{diag}}(d,k)\leqslant|k^\times/k^{\times d}|.$$

*In particular, let $\varphi\cong\langle a_1,\dots,a_m\rangle$ be an anisotropic form of degree $d$ over $k$ such that $m=|k^\times/k^{\times d}|$. Then $\varphi$ is universal or isotropic.*

**Proof.** Let $n = u_{\text{diag}}(d, k)$, then there is an anisotropic diagonal form $\varphi = \langle a_1, \ldots, a_n\rangle$ of degree $d$ over $k$ of dimension $n$. For $n = 1$ there is nothing to show, so assume $n \geqslant 2$. Write $\varphi = \langle a_1\rangle \perp \varphi_1$ with $\varphi = \langle a_2, \ldots, a_n\rangle$, then $D(\varphi_1) \subsetneqq D(\varphi)$ by Proposition 2. Write $\varphi_1 = \langle a_2\rangle \perp \varphi_2$ with $\varphi_2 = \langle a_3, \ldots, a_n\rangle$, then $D(\varphi_2) \subsetneqq D(\varphi_2 \perp \langle a_{n-1}\rangle) \subsetneqq D(\varphi)$ by Proposition 2 and so on. Thus $\varphi$ represents at least $n$ distinct elements of $k^\times/k^{\times d}$, and we get $n \leqslant |k^\times/k^{\times d}|$. $\square$

**Remark 4.** Although Theorem 1 was already proved by Demyanov [De2] in 1956, it seems that this paper, written in Russian and apparently never translated, is mostly unknown to the public. It was proved again for diagonal forms of degree $d$ over nonreal fields as well as for odd degree $d$ over formally real fields by Leep [Le-Y1, 2.3]. Theorem 1 for diagonal forms of degree $d$ over finite fields $\mathbb{F}_q$ can be found in J1, Théorème 1, p. 25] or Small [Sm2, 3.13]). The proofs presented in those papers are all basically identical to the one employed here or in [De2] (1956). For $d = 2$, the result (see [S, p. 105]) was proved by Kneser, see the MathSciNet Review MR0059260 (15,500a) from 1954, and then appeared in the appendix of a paper by Elman and Lam in 1973 [E-La].

## 4. Higher $u$-invariants of valued fields

### 4.1. Discretely valued fields

Let $(k, v)$ be a discretely valued field with valuation ring $R$, maximal ideal $m$, value group $\Gamma$ and residue field $\overline{k} = R/m$. Let $\pi \in R$ be a uniformizing parameter (1.4). $R$ is a principal ideal domain.

**Remark 5.** A similar result as Theorem 1 was given for a complete discrete valuation ring of characteristic 0 with finite residue field, with the obvious definition for $u_{\text{diag}}(d, R)$, in [G, p. 135ff.]. For a complete discrete valuation ring $R$ of characteristic 0 with finite residue field, Theorem 1, using $u_{\text{diag}}(d, R) = u_{\text{diag}}(d, k)$, gives the bound

$$u_{\text{diag}}(d, R) \leqslant |k^\times/k^{\times d}|,$$

which is the same one as obtained in [G, p. 136] for odd $d$ and improves the one in [G, p. 137] for even $d$, since $d|R^\times/R^{\times d}| = |k^\times/k^{\times d}|$. In particular, we have $s_d(R) \leqslant |k^\times/k^{\times d}|$.

As an application of Theorem 1 we obtain:

**Proposition 3.** *Let $k$ be a finite field extension of $\mathbb{Q}_p$ of degree $n$, with valuation ring $R$ and with residue field $\mathbb{F}_q$, where $q = p^f$. Then*

$$u_{\text{diag}}(d, R) \leqslant d \gcd(d, q-1)|\mathbb{Z}_p/\mathrm{d}\mathbb{Z}_p|^e,$$

*where $e$ is the ramification index of $k$ over $\mathbb{Q}_p$. In particular,*

$$u_{\text{diag}}(d, \mathbb{Z}_p) \leqslant d \gcd(d, p-1)|\mathbb{Z}_p/\mathrm{d}\mathbb{Z}_p|.$$

**Proof.** We have $R^\times \cong \mathbb{F}_q^\times \times (1+\pi R)$, and $1+\pi R \cong \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p$ with $e$ copies of $\mathbb{Z}_p$. Hence

$$|R^\times/R^{\times d}| = |\mathbb{F}_q^\times/\mathbb{F}_q^{\times d}||\mathbb{Z}_p/\mathrm{d}\mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p/\mathrm{d}\mathbb{Z}_p|,$$

with $e$ copies of $\mathbb{Z}_p/\mathrm{d}\mathbb{Z}_p$. It is well known that $|\mathbb{F}_q^\times/\mathbb{F}_q^{\times d}| = \gcd(d, q-1)$. Theorem 1 yields the assertion, using that $u_{\mathrm{diag}}(d, R) = u_{\mathrm{diag}}(d, k)$ and $d|R^\times/R^{\times d}| = |k^\times/k^{\times d}|$. □

Let $\varphi$ be a form of degree $d$ and dimension $n$. $\varphi$ is called *normic of order i*, if $\varphi$ is anisotropic and $n = d^i$ [G, p. 16]. Concerning lower bounds for $u(d, R)$, the following result is due to Lang and implies that

$$u(d, R) \geqslant d^{i+1}$$

if there is a normic form $\varphi$ over the residue field $\overline{k}$ of order $i \geqslant 0$:

**Theorem 2** (*cf. Greenberg [G, 4.10]*). *If there is a form $\varphi$ over $\overline{k}$ of degree d which is normic of order $i \geqslant 0$, then there is a form $\Phi$ of degree d over R which is normic of order $i+1$.*

**Proof.** Let $\pi$ be a prime element in $R$. Let $\varphi'$ be a form of degree $d$ over $R$ obtained from $\varphi$ by replacing each coefficient of $\varphi$ by a representative in $R$. Let $v_1, \ldots, v_d$ be independent vectors of $n$ variables each, and consider the form

$$\Phi(v_1, \ldots, v_d) = \varphi'(v_1) + \pi\varphi'(v_2) + \pi^2\varphi'(v_3) + \cdots + \pi^{d-1}\varphi'(v_d).$$

$\Phi$ has degree $d$ and dimension $d^{i+1}$. $\Phi$ has no primitive zero mod $\pi^d$: suppose

$$\Phi(v_1, \ldots, v_d) \equiv 0 \bmod \pi^d \tag{2}$$

with $v_i \in R^n$ for all $i$. Reading this congruence mod $\pi$ gives

$$\varphi'(v_1) \equiv 0 \bmod \pi.$$

Since $\varphi$ is normic of order $i$, $v_1$ is not primitive; i.e., $v_1 = \pi w_1$ for some $w_1 \in R^n$. But then

$$\varphi'(v_1) = \pi^d\varphi'(w_1),$$

and we can divide (2) by $\pi$ which yields

$$\varphi'(v_2) + \pi\varphi'(v_3) + \cdots + \pi^{d-2}\varphi'(v_d) \equiv 0 \bmod \pi^{d-1}.$$

Repeating the above argument we see that $v_2$ is not primitive, and continuing this way, we are able to show that none of the vectors $v_i$ is primitive. □

### 4.2. A theorem for higher $u$-invariants of Henselian valued fields

Let $(k, v)$ be a valued field with valuation ring $R$. Assume that $\mathrm{char}\,\overline{k} \nmid d$.

For $u \in R$, denote by $\bar{u}$ the image of $u$ in $\overline{k}$. For a polynomial $f \in R[X]$, $f = a_n x^n + \cdots + a_1 x + a_0$, define the polynomial $\bar{f} = \bar{a}_n x^n + \cdots + \bar{a}_1 x + \bar{a}_0$ over $\bar{k}$. If $\varphi = \langle a_1, \ldots, a_n\rangle$ is a nondegenerate diagonal form with entries $a_i \in R^\times$, define the diagonal form $\overline{\varphi} =$

$\langle \bar{a}_1, \ldots, \bar{a}_n \rangle$ over $\overline{k}$. $\varphi$ is called a *unit form*, if $\overline{\varphi}$ is nondegenerate. Choose a set $\{\pi_\gamma | \gamma \in I\}$ such that the values of the $\pi_\gamma$'s represent the distinct cosets in $\Gamma/\mathrm{d}\Gamma$. We may decompose a diagonal form $\varphi$ as $\varphi = \perp \varphi'_\gamma$ by taking $\varphi'_\gamma$ to be the diagonal form whose entries comprise all $a_i$ with $v(a_i) = v(\pi_\gamma) \bmod \mathrm{d}\Gamma$. By altering the slots by $d$-powers, if necessary, we may then write $\varphi'_\gamma = \pi_\gamma \varphi_\gamma$ with each $\varphi_\gamma$ a diagonal unit form. There are only finitely many non-trivial $\varphi_\gamma$ [Mo].

If $\Gamma = \mathbb{Z}$, the set $\{\pi_\gamma | \gamma \in I\}$ can be chosen to be $\{\pi^i | i = 0, \ldots, d-1\}$ and $|\Gamma/\mathrm{d}\Gamma| = d$ is finite.

**Hensel's Lemma.** [Ribenboim [R, p. 13]] *Let $f \in R[X]$, $u \in R$ and $\bar{f} = (x - \bar{u})h$, where $h \in \bar{k}[X]$ with $\deg(h) \neq 0$, $h(\bar{u}) \neq 0$. Then there exists an element $u' \in R$ such that $\overline{u'} = \bar{u}$ and $f(u') = 0$.*

$(k, v)$ is called a *Henselian valued field* and $R$ a *Henselian valuation ring* if Hensel's Lemma is satisfied by the ring $R$. For a survey of different formulations of Hensel's Lemma, the reader is referred to [R]. Every complete discretely valued field is Henselian.

We obtain, by applying Hensel's Lemma, a special case of [Mo, Theorem 2.5]:

**Theorem 3** (*Morandi [Mo, Proposition 3.1]*). *Let $(k, v)$ be a Henselian valued field. Let $\varphi$ be a diagonal form. Write $\varphi = \pi_1 \varphi_1 \perp \cdots \perp \pi_r \varphi_r$ with each $\varphi_i$ a diagonal unit form and the $\pi_i$ having distinct values in $\Gamma/\mathrm{d}\Gamma$. Then $\varphi$ is isotropic if and only if some $\overline{\varphi_i}$ is isotropic.*

Among other things, its proof uses the following observation (a special case of [Mo, 2.3]) on non-trivial zeroes of nondegenerate diagonal forms:

**Lemma 4.** *Let $(k, v)$ be a Henselian valued field. For a diagonal unit form $\varphi$, $\varphi$ is isotropic if and only if $\overline{\varphi}$ is isotropic.*

**Proof.** One direction is obvious: If $(a_1, \ldots, a_n)$ is an isotropic vector of $\varphi$, we scale it to assume each $a_i \in R$ and some $a_j \in R^\times$. Then $(\overline{a_1}, \ldots, \overline{a}_n)$ is an isotropic vector of $\overline{\varphi}$.

Conversely, we observe that since $\overline{\varphi_i}$ is a diagonal form and $\operatorname{char} \bar{k}$ does not divide $d$, it is easy to check that any non-trivial zero $\alpha = (\alpha_1, \ldots, \alpha_n)$ of $\overline{\varphi_i}(x_1, \ldots, x_n)$ is nonsingular, i.e. $\partial f/\partial x_i(\alpha) \neq 0$ for some $i$.

Choose $u_i \in R$ with $\bar{u}_i = \alpha_i$ for each $i$ and consider the polynomial

$$h(t) = \varphi(u_1, \ldots, u_{i-1}, t, u_{i+1}, \ldots, u_n).$$

Then $\bar{h}(\alpha_i) = 0$ and $(\mathrm{d}\bar{h}/\mathrm{d}t)(\alpha_i) = (\partial\overline{\varphi}/\partial x_i)(\alpha) \neq 0$. Hensel's Lemma implies that there is an element $w \in R$ such that $h(w) = 0$. Thus $(u_1, \ldots, u_{i-1}, w, u_{i+1}, \ldots, u_n)$ is an isotropic vector for $\varphi$ and is non-zero since it is a lift of $\alpha$. $\square$

**Theorem 4.** *Let $(k, v)$ be a Henselian valued field with valuation ring $R$, value group $\Gamma$ and residue field $\overline{k}$. Assume $\operatorname{char} \overline{k} \nmid d$.*

(i) *If both $|\Gamma/\mathrm{d}\Gamma|$ and $u_{\mathrm{diag}}(d, \overline{k})$ are finite, then*

$$u_{\mathrm{diag}}(d, k) = |\Gamma/\mathrm{d}\Gamma| \, u_{\mathrm{diag}}(d, \overline{k}).$$

(ii) *$u_{\mathrm{diag}}(d, k)$ is infinite if either $|\Gamma/\mathrm{d}\Gamma|$ is infinite or $u_{\mathrm{diag}}(d, \overline{k})$ is infinite.*

**Proof.** Choose a set $\{\pi_\gamma | \gamma \in I\}$ such that the values of the $\pi_\gamma$'s represent the distinct cosets in $\Gamma/\mathrm{d}\Gamma$.

(i) Let $u_{\mathrm{diag}}(d,\overline{k})$ and $|\Gamma/\mathrm{d}\Gamma|$ both be finite. Choose an anisotropic diagonal form $\varphi$ over $\overline{k}$. Lift $\varphi$ to a form $\tilde{\varphi}$ over $k$. The form

$$\phi = \perp_{\gamma\in I}\pi_\gamma\tilde{\varphi}$$

is anisotropic over $k$ by Theorem 3, hence $u_{\mathrm{diag}}(d,k)\geqslant|\Gamma/\mathrm{d}\Gamma|u_{\mathrm{diag}}(\overline{k})$. Let $\varphi$ be an anisotropic diagonal form of degree $d$ over $k$. Decompose $\varphi$ as $\varphi = \perp \pi_\gamma\varphi_\gamma$ with each $\varphi_\gamma$ a diagonal unit form. Let $m$ be the number of those finitely many $\varphi_\gamma$. Since $\varphi$ is anisotropic, all nonzero $\overline{\varphi_\gamma}$ must be anisotropic over $\overline{k}$ by Theorem 3. Therefore we conclude that $u(d,\overline{k})\geqslant \dim\overline{\varphi}_\gamma$ for all nonzero $\overline{\varphi}_\gamma$. Since each $\overline{\varphi}_\gamma$ is also a diagonal form, we have moreover that $u_{\mathrm{diag}}(d,\overline{k})\geqslant \dim\overline{\varphi_\gamma}$ for all nonzero $\overline{\varphi}_\gamma$. Hence if $u_{\mathrm{diag}}(d,k)=\dim\varphi$, we conclude that $u_{\mathrm{diag}}(d,k)=\sum\dim\overline{\varphi}_\gamma\leqslant m u_{\mathrm{diag}}(d,\overline{k})$, where $m$ is the number of the $\varphi_\gamma$ which are nonzero. Indeed, we have $m\leqslant|\Gamma/\mathrm{d}\Gamma|$. Thus $u_{\mathrm{diag}}(d,k)\leqslant|\Gamma/\mathrm{d}\Gamma|u_{\mathrm{diag}}(\overline{k})$.

(ii) If $\Gamma/\mathrm{d}\Gamma$ is an infinite group, we can take arbitrarily many $\pi_\gamma$ to obtain an anisotropic form of arbitrarily large dimension by the construction above, which shows that $u_{\mathrm{diag}}(d,k)$ is infinite. A similar argument using a unit form, which is the lift of an anisotropic form over $\overline{k}$ shows that $u_{\mathrm{diag}}(d,k)$ is infinite when $u_{\mathrm{diag}}(d,\overline{k})$ is infinite. □

For $d=2$, Theorem 4 corresponds to Springer's theorem for quadratic forms over Henselian valued fields [Sp2]. For a similar result for quadratic forms, see also [Du].

**Remark 6.** According to MathSciNet review MR0037836 (12,315d) of [De1], Demyanov proved that

$$u(3,k)\leqslant 3u(3,\overline{k})$$

for any field $k$, which is complete under a discrete valuation with residue class field $\overline{k}$ of characteristic not 3. This result was also proved by Springer [Sp1], this time including the case that $\mathrm{char}\,\overline{k}$ is 3. It could be seen as an indication for the existence of a more general version of Theorem 4 for cubic forms (or even of forms of degree higher than 3), which are not necessarily diagonal. The methods Springer uses in his proof do not generalize to degree greater than 3, however [Sp1, Section 4].

**Corollary 2.** (i) *Let $(k,v)$ be a discretely valued field with valuation ring $R$ and residue field $\overline{k}$. Assume $\mathrm{char}\,\overline{k}\nmid d$. Then*

$$u_{\mathrm{diag}}(d,k)\geqslant d u_{\mathrm{diag}}(d,\overline{k}).$$

(ii) *Let $k$ be any field of $\mathrm{char}\,k\nmid d$. Let $l/k$ be a field extension of finite type over $k$ of transcendence degree $n$. Then*

$$u_{\mathrm{diag}}(d,l)\geqslant d^n u_{\mathrm{diag}}(d,k')$$

*for a suitable finite field extension $k'/k$.*

**Proof.** (i) Let $u_{\text{diag}}(d, \overline{k})$ be finite. Choose an anisotropic form $\varphi$ over $\overline{k}$. Lift $\varphi$ to a form $\tilde{\varphi}$ over $k$. Then $\phi = \perp_{\gamma\in I} \pi_\gamma \tilde{\varphi}$ is anisotropic over $k$, since it is anisotropic over the completion of $k$ by Theorem 3. If $u_{\text{diag}}(d, \overline{k})$ is infinite, the same argument as in Theorem 4(ii) yields the assertion.

(ii) Take $n = 1$. Write $l$ as finite extension of $k(x)$. The discrete valuation of $k(x)$ given by the ideal $(x)$ of $k[x]$ extends to a discrete valuation of $l$ with residue class field $k'$, which is a finite extension of $k$. By (i), $u_{\text{diag}}(d, l) \geqslant d u_{\text{diag}}(d, k')$. The assertion now follows by induction on $n$. □

**Remark 7.** (i) Let $k$ be a $C_i$-field and let $K = k((x))$ be the field of formal Laurent series in $x$ over $k$. Then $K$ is a $C_{i+1}$-field [Pf1, Chapter 5, 2.2].

As in [Pf1, p. 111] for $d = 2$, we can deduce from Theorem 4 – without Tsen–Lang theory – that the iterated power series field $K = k((x_1)) \dots ((x_n))$ over a field $k$ where $\operatorname{char} k \nmid d$, $1 \leqslant n \leqslant \infty$, has $u_{\text{diag}}(d, K) = d^n u_{\text{diag}}(d, k)$. In particular if $k$ is algebraically closed, we have $u_{\text{diag}}(d, K) = d^n$.

(ii) Let $m_p = u_{\text{diag}}(d, \mathbb{F}_p)$. Then

$$u_{\text{diag}}(d, \mathbb{F}_p((t_1)) \cdots ((t_n))) = d^n m_p$$

by (i) for all $p$ with $p \nmid d$ and

$$u(d, \mathbb{F}_p((t_1)) \cdots ((t_n))) \leqslant d^{n+1}$$

by Tsen–Lang theory for all $p$ [G, (4.8)]. Since $u_{\text{diag}}(d, \mathbb{F}_p) \leqslant d$ by Chevalley,

$$u_{\text{diag}}(d, \mathbb{F}_p((t_1)) \cdots ((t_n))) = d^n m_p \leqslant d^{n+1}$$

for all $p$ with $p \nmid d$.

## 5. On $u$-invariants of finite fields, $p$-adic fields and $p$-adic rational function fields

### 5.1. Finite fields

Let $k = \mathbb{F}_q$ with $q = p^s$, $p$ prime. Chevalley's well-known theorem that every finite field is a $C_1$-field [S, p. 97] implies that

$$u(d, \mathbb{F}_q) \leqslant d.$$

Orzech [O] proved the slightly stronger bound

$$u_{\text{diag}}(d, \mathbb{F}_q) \leqslant d - 1 \quad \text{if } -1 \in \mathbb{F}_q^{\times d} \text{ and } d \geqslant 4.$$

Let $d^* = \gcd(d, q-1)$. Then $|\mathbb{F}_q^\times / \mathbb{F}_q^{\times d}| = |\mathbb{F}_q^\times / \mathbb{F}_q^{\times d^*}| = d^*$, so that

$$u_{\text{diag}}(d, \mathbb{F}_q) = u_{\text{diag}}(d^*, \mathbb{F}_q) \leqslant d^*$$

by Theorem 1. In particular, $s_d(\mathbb{F}_q) = u_{\text{diag}}(d, \mathbb{F}_q) = 1$, if $d$ is relatively prime to $q - 1$.

For $q > (d^* - 1)^2$, every element of $\mathbb{F}_q$ is a sum of two $d$th powers [Sm2, p. 148]. The proofs given in [Sm2] show that in this case the form $n \times \langle 1 \rangle = \langle 1, \ldots, 1 \rangle$ becomes universal for some $n$. Indeed, if $q > (d^* - 1)^4$ then $u_{\mathrm{diag}}(d, \mathbb{F}_q) = 2$ [Sm1].

Any element in a finite field $\mathbb{F}_q$, which is a sum of $d$th powers must be a sum of $d$, $d$th powers (Tornheim's Theorem [Sm2, 3.16]). As soon as $q$ is "large enough" with respect to the exponent $d$, every element in $\mathbb{F}_q$ is a sum of two $d$th powers [Sm2, 6.12]. For instance, every element in $\mathbb{F}_q$ is a sum of two 4th powers provided $q > 41$.

### 5.2. $p$-Adic fields

Due to Artin's conjecture that $u(d, \mathbb{Q}_p) \leqslant d^2$, $p$-adic fields attracted a lot of attention. The conjecture was verified for $d = 2$ by Hasse, for $d = 3$ independently by Lewis [L] and Demyanov [De1], and for $d = 5, 7$ and 11, under the assumption that $q$ is large enough, by Birch–Lewis [Bi-L], Laxton–Lewis [La-L] (see also Knapp [Kn]) and Leep–Yeomans [Le-Y2]. Lower bounds for the size of $q$ in these cases were given in [Kn] and [Le-Y2]. Although the conjecture turned out to be wrong, see [T] or later Lewis and Montgomery [L-Mon] for counterexamples, it is believed to be true for primes.

**Example 2** (*communicated by D. Leep*). Since the field $\mathbb{F}_p$ has an algebraic extension of degree $d$, there exists a homogeneous form $\varphi(x_1, \ldots, x_d) \in \mathbb{Z}[x_1, \ldots, x_d]$ of degree $d$ in $d$ variables, which is anisotropic modulo $p$. The form $\Phi = \langle 1, p, \ldots, p^{d-1} \rangle \otimes \varphi$ is of degree $d$ in $d^2$ variables with coefficients in $\mathbb{Z}$. Since $\varphi$ is anisotropic modulo $p$, $\Phi$ must be anisotropic over $\mathbb{Q}_p$. This implies that $\Phi$ is anisotropic over $\mathbb{Q}$. Thus

$$u(d, \mathbb{Q}_p) \geqslant d^2 \quad \text{and} \quad u(d, \mathbb{Q}) \geqslant d^2.$$

This also follows directly from Theorem 2.

The most up-to-date general results on upper bounds seem to be the ones of Wooley [W] from 1998, who, for instance, proves the estimate

$$u(d, \mathbb{Q}_p) \leqslant d^{2^d}.$$

In 2007, Heath-Brown [H-B2] proved the much stronger bounds

$$u(4, \mathbb{Q}_2) \leqslant 9142,$$

$$u(4, \mathbb{Q}_p) \leqslant 128 \quad \text{for } p = 3 \text{ and } p = 7,$$

$$u(4, \mathbb{Q}_5) \leqslant 312,$$

$$u(4, \mathbb{Q}_p) \leqslant 120 \quad \text{for } p \geqslant 11$$

for quartic forms, and that

$$u(5, \mathbb{Q}_p) \leqslant 25 \quad \text{if } p \geqslant 17.$$

**Remark 8.** Using the Ax–Kochen–Ersov transfer theorem from the model theory of valued fields [Ax-K], Ax and Kochen proved that given a degree $d$, for almost all primes $p$, a form of degree $d$ over $\mathbb{Q}_p$ of dimension greater than or equal to $d^2+1$ will be isotropic [G, (7.4)]. (So Artin's conjecture would be almost true in that sense.)

Let $k$ be a finite field extension of $\mathbb{Q}_p$ with residue class field $\overline{k}=\mathbb{F}_q$. We assume that char $\mathbb{F}_q = p\nmid d$ to be able to apply Theorem 4, which yields

$$u_{\text{diag}}(d,k)=du_{\text{diag}}(d,\mathbb{F}_q).$$

If $d$ is relatively prime to both $p$ and $q-1$ (hence odd), then by 5.1,

$$d=u_{\text{diag}}(d,k).$$

Let $d^*=\gcd(d,q-1)$. Then $u_{\text{diag}}(d,\mathbb{F}_q)=u_{\text{diag}}(d^*,\mathbb{F}_q)$, see 5.1, and Theorem 1 yields

$$d\leqslant u_{\text{diag}}(d,k)=du_{\text{diag}}(d,\mathbb{F}_q)\leqslant dd^*$$

(since the $d$th power level of $k$ is finite [G, (7.18)]).

In particular, the estimate

$$u_{\text{diag}}(d,k)=du_{\text{diag}}(d,\mathbb{F}_q)\leqslant d^2$$

is recovered, which has been known for some time, see [Al].

If $d\geqslant 4$ and $-1\in\mathbb{F}_q^{\times d}$ then $u_{\text{diag}}(d,\mathbb{F}_q)\leqslant d-1$ by 5.1, hence

$$u_{\text{diag}}(d,k)=du_{\text{diag}}(d,\mathbb{F}_q)\leqslant d(d-1)=d^2-d.$$

Moreover, $s_{p-1}(\mathbb{F}_p)=p-1$ for any $p\neq 2$, which implies $u_{\text{diag}}(p-1,\mathbb{F}_p)=p-1$, since $p-1\leqslant u_{\text{diag}}(p-1,\mathbb{F}_p)\leqslant \gcd(p-1,p-1)=p-1$. Thus

$$u_{\text{diag}}(p-1,\mathbb{Q}_p)=(p-1)^2.$$

So the upper bound $u_{\text{diag}}(d,\mathbb{Q}_p)\leqslant d^2$ given by Joly [J2, p. 97] for any $p\neq 2$ is best possible. Furthermore,

$$(p-1)^2\leqslant u(p-1,\mathbb{Q}_p).$$

For $p=2$ and $d=2^r$, Joly [J2, p. 97] proved that

$$u_{\text{diag}}(d,\mathbb{Q}_p)\leqslant 2d^2.$$

**Example 3.** Let $k$ be a finite field extension of $\mathbb{Q}_p$. We can use Demyanov's upper bound from Theorem 1 again to obtain upper bounds for $u_{\text{diag}}(d,k)$, which supersede the ones found by Alemu [Al]: Since the $d$th power level of $k$ is finite [G, (7.18)], by Theorem 1 we have

$$u_{\text{diag}}(d,k)\leqslant \frac{d}{|d|_p}w,$$

where $w$ is the number of $d$th roots of unity contained in $k$ and $|d|_p=1/p^{ord_p(d)}$ with $ord_p(d)$ being the highest power of $p$ dividing $d$ [Ko, p. 73]. If $d$ is not divisible by $p$ and $k$ contains no $d$th roots of unity other than 1 itself, then even

$$u_{\text{diag}}(d,k)\leqslant d.$$

This greatly improves the bound in [Al], which depends on the degree $n = [k : \mathbb{Q}_p]$:

$$u_{\rm diag}(d, k) < \max(3nd^2 - nd + 1, 2d^3 - d^2)$$

if $p > 2$ divides $d$, and

$$u_{\rm diag}(d, k) < 4nd^2 - nd + 1$$

if $p = 2$.

### 5.3. $p$-Adic rational function fields

It is a long-standing question whether the finiteness of the classical $u$-invariant $u(k)$ of quadratic forms of a field $k$ of characteristic not 2 implies the finiteness of $u(k(t))$.

For $k$, a non-dyadic $p$-adic field, this was proved by Hoffmann-van Geel [Ho-vG] and independently by Merkurjev [M3]. After lowering the bound for the $u$-invariant of a function field $K$ of transcendence degree one over a non-dyadic $p$-adic field from $u(2, K) \leqslant 22$ [Ho-vG] to $u(2, K) \leqslant 10$ [Pa-S1], Parimala and Suresh [Pa-S2] proved that

$$u(2, K) = 8$$

in 2007.

Using a completely different approach, i.e. the Ax–Kochen–Ersov transfer theorem from the model theory of valued fields [Ax-K], Zahidi [Z] had shown in 2005 that any quadratic form over $\mathbb{Q}(t_1, \ldots, t_n)$ of dimension greater than $2^{n+2}$ is isotropic over the field $\mathbb{Q}_p(t_1, \ldots, t_n)$ for almost all primes $p$. This result resembles the one by Ax–Kochen, see Remark 8. The proof of [Z] easily extends to our setting of higher degree forms and yields:

**Theorem 5.** *Let $\varphi$ be a form of degree $d \geqslant 2$ over $\mathbb{Q}(t_1, \ldots, t_n)$ of dimension greater than $d^{n+2}$. Then $\varphi$ is isotropic over the rational function field $\mathbb{Q}_p(t_1, \ldots, t_n)$ for almost all primes $p$.*

This result seems not to have appeared in the literature so far.

By Corollary 1,

$$u_{\rm diag}(d, \mathbb{Q}_p(t_1, \ldots, t_n)) \geqslant d^n u_{\rm diag}(d, \mathbb{Q}_p)$$

and

$$u(d, \mathbb{Q}_p(t_1, \ldots, t_n)) \geqslant d^n u(d, \mathbb{Q}_p).$$

Put $m_p = u_{\rm diag}(d, \mathbb{F}_p)$. If $p \nmid d$, then $u_{\rm diag}(d, \mathbb{Q}_p) = dm_p$ by Theorem 4. One may conjecture that for all primes $p$ with $p \nmid d$,

$$u_{\rm diag}(d, \mathbb{Q}_p(t_1, \ldots, t_n)) = d^{n+1} m_p.$$

## References

[Al] Y. Alemu, On zeros of diagonal forms over $p$-adic fields, Acta Arith. 48 (3) (1987) 261–273.

[Am-K] Y. Amice, B. Kahn, Sommes de puissances dans les corps finis, Université Paris 7, Unité de Recherche Associéee 212 - Theories Géeometriques Nr. 28, April 1992.

[A] E. Artin, The collected papers of Emil Artin, Addison-Wesley, Reading, MA, 1965.

[Ax-K] J. Ax, S. Kochen, Diophantine problems over local fields I, Amer. J. Math. 87 (1965) 605–630.

[B-C] E. Becker, M. del. P. Canales, Higher levels of $\mathbb{Z}/p^l\mathbb{Z}$ and period equations, Manuscripta Math. 99 (1999) 425–441.

[Bi-L] B.J. Birch, D.J. Lewis, *P*-Adic forms, J. Indian Math. Soc. (N.S.) 23 (1959) 11–32.

[D] H. Davenport, Cubic forms in 16 variables, Proc. R. Soc. London Ser. A 272 (1963) 285–303.

[De1] V.B. Demyanov, On cubic forms in discretely normed fields, Dokl. Akad. Nauk SSSR 74 (1950) 889–891.

[De2] V.B. Demyanov, On representation of a zero of forms of the form $\sum a_i X_i^n$, (Russian), Dokl. Akad. Nauk SSSR 105 (1955) 203–205.

[Du] W.H. Durfee, Quadratic forms over fields with a valuation, Bull. Am. Math. Soc. 54 (1948) 338–351.

[E-La] R. Elman, T.Y. Lam, Quadratic forms and the *u*-invariant. I, Math. Z. 131 (1973) 283–304.

[G] M.J. Greenberg, Lectures on Forms in Many Variables, W.A. Benjamin, Inc., New York, Amsterdam, 1969.

[H-P] D.K. Harrison, B. Pareigis, Witt rings of higher degree forms, Comm. Alg. 16 (6) (1988) 1275–1313.

[H-B1] D.R. Heath-Brown, Cubic forms in ten variables, Proc. London Math. Soc. (3) 47 (2) (1983) 225–257.

[H-B2] D.R Heath-Brown, Zeros of *p*-adic forms, Preprint, 2008.

[Ho-vG] D. Hoffmann, J. Van Geel, Zeros and norm groups of quadratic forms over function fields in one variable over a local non-dyadic field, J. Ramanujan Math. Soc. 13 (1998) 85–110.

[I-R] K. Ireland, M. Rosen, A Classical Introduction to Modern Number Theory, second ed., Springer, Berlin, Heidelberg, New York, 1990.

[J1] J.-R. Joly, Équations et variétés algébriques sur un corps fini, L'Enseignement Math. 19 (1973) 1–117.

[J2] J.-R. Joly, Sommes de puissances *d*-ièmes dans un anneau commutatif, Acta Arith. (1970) 37–114.

[Kn] M.P. Knapp, Artin's conjecture for forms of degree 7 and 11, J. London Math. Soc. (2) 63 (2) (2001) 268–274.

[KMRT] M.A. Knus, A. Merkurjev, M. Rost, J.-P. Tignol, The Book of Involutions, vol. 44, AMS Coll. Publications, 1998.

[Ko] N. Koblitz, *p*-Adic Numbers, *p*-Adic Analysis and Zeta-Functions, Springer, Berlin, Heidelberg, New York, 1977.

[La] T.Y. Lam, The Algebraic Theory of Quadratic Forms, Benjamin, Reading, MA, 1973 (revised printing 1980).

[La-L] R.R. Laxton, D.J. Lewis, Forms of degrees 7 and 11 over *p*-adic fields, in: Proceedings of Symposium on Pure Mathematics, vol. VIII, AMS, Providence, RI, 1965, pp. 16–21.

[Le-S] D.B. Leep, W.M. Schmidt, Systems of homogeneous equations, Invent. Math. 71 (1983) 539–549.

[Le-Y1] D.B. Leep, Yeomans, C. Charles, Diagonal forms over fields with procyclic Galois group, Comm. Alg. 20 (7) (1992) 2029–2041.

[Le-Y2] D.B. Leep, Yeomans, C. Charles, Quintic forms over *p*-adic fields, J. Number Theory 57 (2) (1996) 2031–2041.

[L] D.J. Lewis, Cubic homogeneous polynomials over *p*-adic number fields, Ann. Math. 56 (2) (1952) 473–478.

[L-Mon] D.J. Lewis, H.L. Montgomery, On zeros of *p*-adic forms, Mich. Math. J. 30 (1983) 83–87.

[M1] A. Merkurjev, Kaplansky's conjecture in the theory of quadratic forms, (Russian), Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI) 175 (1989), no. Koltsa i Moduli 3, 75–89, 163–164.

[M2] A. Merkurjev, Simple algebras and quadratic forms, (Russian), Izv. Akad. Nauk SSSR Ser. Mat. 55 (1) (1991) 218–224.

[M3] A. Merkurjev, Division algebras over *p*-adic curves, Report on Saltman's Work, Lectures at UCL, June 1997, handwritten notes by J.-P. Tignol.

[Mo] P. Morandi, Springer's theorem for higher degree forms, Math. Z. 256 (1) (2007) 221–228.

[O] M. Orzech, Forms of low degree in finite fields, Bull. Aust. Math. Soc. 30 (1) (1984) 45–58.

[P-R] S. Pall, A.R. Rajwade, Power stufe of Galois fields, Bull. Soc. Math. Belg. 35 (1983) 123–130.

[Pa-S1] R. Parimala, V. Suresh, Isotropy of quadratic forms over function fields of *p*-adic curves, Inst. Hautes Etudes Sci. Publ. Math. 88 (1998) 129–150.

[Pa-S2] R. Parimala, V. Suresh, The *u*-invariant of the function fields of *p*-adic curves, Preprint, arXiv:math.NT/0708.3128v1, 2007.

[P-A-R] J.C. Parnami, M.K. Agrawal, A.R. Rajwade, On the 4th power stufe of a field, Rend. Circ. Mat. Palermo 30 (2) (1981) 245–254.

[Pf1] A. Pfister, Quadratic Forms with Applications to Algebraic Geometry and Topology, Cambridge University Press, London, 1995.

[Pf2] A. Pfister, Quadratische formen in beliebigen körpern, Inv. Math. 1 (1996) 116–132.
[R] P. Ribenboim, Equivalent forms of Hensel's lemma, Expo. Math. 3 (1985) 3–24.
[S] W. Scharlau, Quadratic and Hermitian Forms, Springer, Berlin, Heidelberg, New York, 1985.
[Sm1] C. Small, Sums of powers in large finite fields, Proc. Am. Math. Soc. 65 (1977) 35–36.
[Sm2] C. Small, Arithmetic of Finite Fields, Marcel Dekker, Inc., New York, Basel, Hong Kong, 1991.
[Sp1] T.A. Springer, Some properties of cubic forms over fields with a discrete valuation, Nederl. Acad. Wetensch. Proc. Ser. A 58 = Indag. Math. 17 (1955) 512–516.
[Sp2] T.A. Springer, Quadratic forms over a field with a discrete valuation, Indag. Math. 17 (1979) 33–39.
[T] G. Terjanian, Un contre-exemple à une conjecture d'Artin, C. R. Acad. Sci. Paris Sér. A-B 262 (1966) A612.
[W] T.D. Wooley, On the local solubility of diophantine systems, Compos. Math. 111 (1998) 149–165.
[Z] K. Zahidi, On the $u$-invariant of $p$-adic function fields, Comm. Alg. 33 (7) (2005) 2307–2314.